Quantum Chaos and Complexity

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in collaboration with
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Classical systems with few degrees of freedom

A) **1 degree of freedom** + arbitrary potential V

Hamiltonian $H = \frac{p^2}{2} + V_1(x)$

System **integrable** for any potential $V_1(x)$

$\Rightarrow$ long time trajectory prediction possible

B) **2 (or more) degrees of freedom** + a generic interaction

Hamiltonian $H = \frac{p_x^2 + p_y^2}{2} + V_{12}(x, y)$

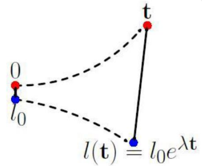
System **non integrable** for a typical potential $V_{12}(x, y) \neq V_1(x)V_2(y)$

$\Rightarrow$ long time trajectory prediction **impossible**

generically **chaotic** dynamics:

sensitivity on initial conditions,

positive Lyapunov exponent $\Lambda > 0$



Transition to chaos & ‘complex’ dynamics

Chirikov standard map (kicked rotator)

Consider $2D$ **standard map** on the torus $x, p \in [0, 2\pi)$

$x' = x + p$

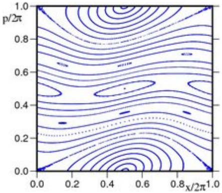
$p' = p + K \cos x'$

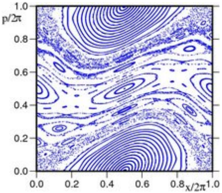
for $K = 0$ the dynamics is **regular** (rotations)

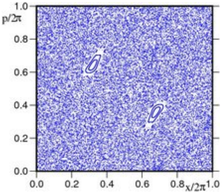
for $K > 0$ unstable trajctories appear and

for $K > K_c \approx 0.971635$ last **KAM tori** are broken

and chaotic layers are connected.







Kicking strength $K = 0.5$ $K \approx K_c = 0.9716$ $K = 5.0$

Chirikov standard map (kicked rotator) II

Consider $2D$ **standard map** on the torus $x, p \in [0, 2\pi)$

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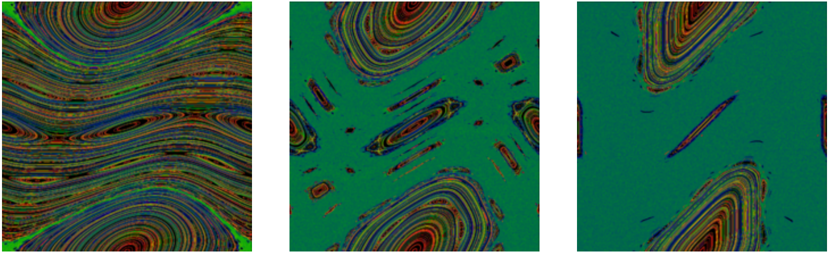


Kicking strength $K = 0.6$ $K = 1.2$ $K = 2.0$

Deterministic chaos and predictability

Chaotic system f is **deterministic**, but its behavior cannot be predicted for times larger than Lyapunov time $T_{\rm Lap} = 1/\Lambda$

Here $$\Lambda = \lim_{t\to\infty} \lim_{\delta x\to 0} \frac{1}{t} \ln \frac{|f^t(x) - f^t(x+\delta x)|}{\delta x}$$

denotes the maximal (local) Lyapunov exponent
(which depends on position x in the phase space)

examples of **Lyapunov time** $T_{\rm Lap}$:

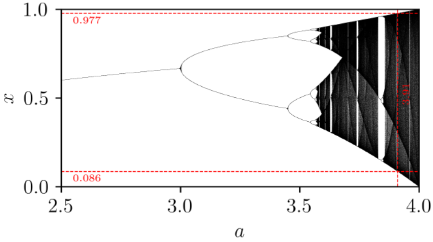
solar system – 5 000 000 years
rotation of Hyperion (moon of Saturn) – 36 days
chemical oscillations – 5 minutes
hydrodynamic oscillations – 2 seconds
1 cm^3 of argon at room temperature – 10^{-11} sec.

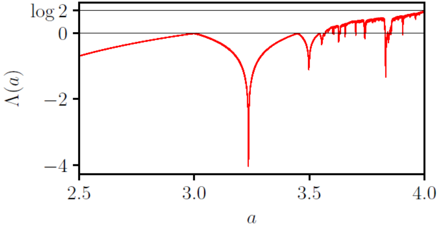
The shorter **Lyapunov time** $T_{\rm Lap}$, the more unpreditable and complex dynamics....

limiting case $T_{\rm Lap} \to 0$ describes **random** sequences

logistic map: $x' = ax(1-x)$

bifurcation diagram of **logistic map** as a function of parameter a





and the corresponding Lyapunov exponent $\Lambda(a)$

several facets of *complexity*

Disorganised/organized Complexity – **Warren Weaver** (1948)

a) **disorganized complexity** – large (say, more than 10^6) subsystems imply random behaviour (gas in a container)

b) **organized complexity** – correlated interaction between small number of subsystems (living systems - *self-organized complexity*).

computational complexity of an algorithm – minimal resources (time, memory) to execute it expressed as a function of the input size n –
Manuel Blum (1967)

descriptive (Kolmogorov) complexity of an object (tex, figure) is the length of the shortest computer program that produces such an output.
Ray Solomonoff (1960), **Andrey Kolmogorov** (1963), **Gregory Chaitin** (1966-68)

example: a **random** sequence of length n bits
has to be written down so it requires n bits

nonlinear dynamics: Example I (2000)

Is this figure complex?

nonlinear dynamics: Example I (2000)

Is this figure complex?

not is sense of **Kolmogorov complexity** !

invariant set of a parabolic map on a square

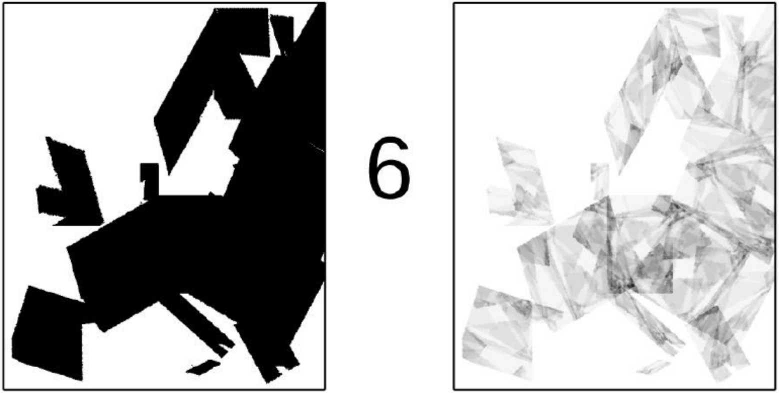
$x' = x + \exp(y - x) - 1|_{\mathrm{mod}1}$

$y' = y + \exp(y - x) - 1|_{\mathrm{mod}1}$

Ashwin, Fu, Nishikawa, K.Ż, *Nonlinearity* 2000

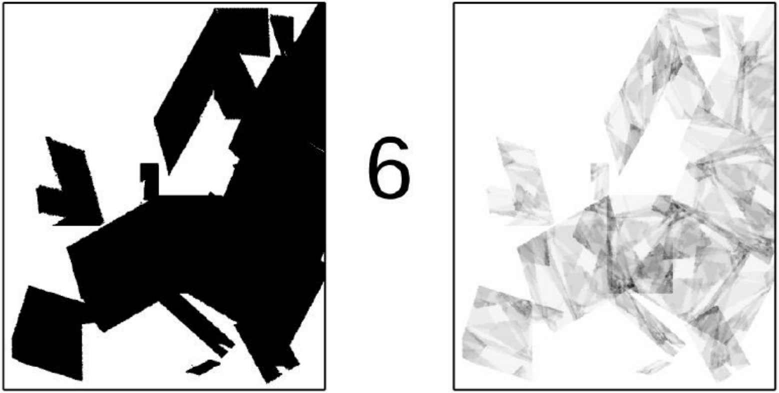
‘complex’ dynamics: Example II (April 16, 2003)

Is this figure complex?



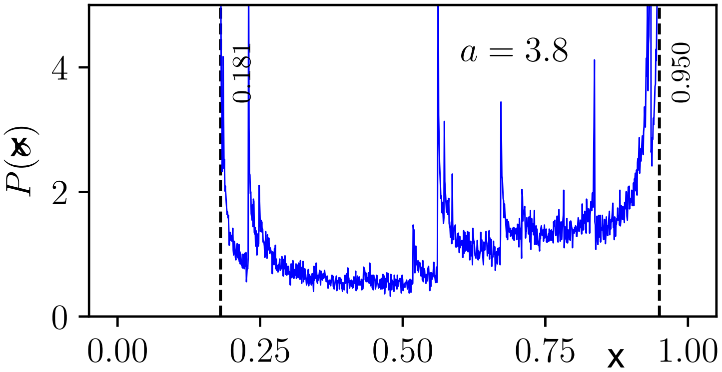
‘complex’ dynamics: Example II (April 16, 2003)

Is this figure complex?



not quite ... 6-th iteration of an *iterated function system* (IFS) $\{F_i(x,y),\ p_i = 1/13\}_{i=1}^{13}$ defined by 13 linear maps F_i acting on $[0,1]^2$ approximating **invariant set** and invariant measure, A. Łoziński, K.Ż, 2003 logistic map: $x' = ax(1-x)$ shows **chaotic dynamics**

for $a > a_c \approx 3.5699$ (apart of periodic windows)

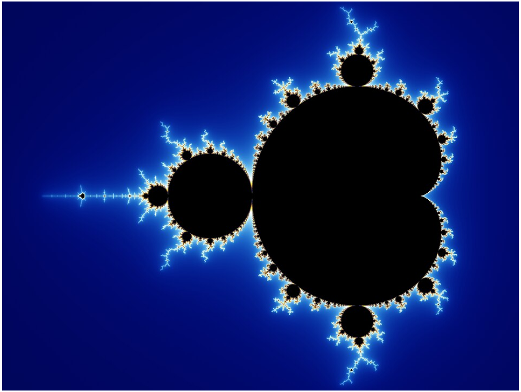


invariant density $P(x)$ for **logistic map** for $a = 3.8$ (blue curve)

displays fractal properties

complexity and chaos – Mandelbrot set

contains **complex** numbers c,
for which the function, $f_c(z) = z^2 + c$, *does not* diverge to infinity



dynamics, related to *logistic map*, $x' = ax(1-x)$,
was defined and drawn by **Robert Brooks** and **Peter Matelski** (1978),
popularized by **Benoit Mandelbrot** who discovered self-similarity (1980).

recap: classical chaos, randomness & complexity

a) **chaotic dynamical systems** (positive Lyapunov exponent $\Lambda > 0$, positive Kolmogorov-Synai dynamical entropy $H_{KS} > 0$), produce time series which look *apparently random*, but they will not pass all the statistical tests for randomness.

b) deterministic time sequence seems to be **complex**,
but its **Kolmogorov complexity** is low.

c) any true random sequence has high **Kolmogorov complexity** –
to encode it it one needs to write down entire sequence.

Anyone who considers arithmetical methods
of producing random digits is, of course,
in a state of sin.

John von Neumann

with *deterministic means* one generates psuedo-random numbers only...


beeline

Otton Nikodym & Stefan Banach,

talking at a bench in Planty Garden, **Cracow, summer 1916**

Quantum systems - a non-relativistic approach

A simple (closed) quantum system S_1 described by

a Hamiltonian H_1 in a **finite dimensional** complex Hilbert space $\mathcal{H}_N$.
Unitary dynamics, $U(t) = \exp(-iH_1 t)$

classically **regular dynamics** $\Rightarrow$ structured Hamiltonian matrix H_1

A composed quantum system S_{1+2} described by

a Hamiltonian H_{12} in a complex Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$.

classically **chaotic dynamics** $\Rightarrow$ structureless Hamiltonian H_{12}
described by a **random hermitian matrix**

(from a suitable ensemble).

Random matrices: applications in quantum physics

Quantum Chaos and Unitary Dynamics:

'Quantum chaology' - Michael Berry 1987

Quantum analogues of classically chaotic dynamical systems can be described by **random matrices**

a) autonomous systems – hermitian **Hamiltonians**:
Gaussian ensembles of random Hermitian matrices,
(GOE, GUE, GSE)
example - **coupled spins**

b) periodic systems – unitary **evolution operators**:
Dyson circular ensembles of random unitary matrices,
(COE, CUE, CSE)
example - **quantum kicked rotator**

Gaussian Ensembles of Hermitian matrices H

Hermitian random matrix $H = H^\dagger$

consists of **independent Gaussian** entries

a) **orthogonal ensemble**, $\beta = 1$ - real random numbers,

b) **unitary ensemble**, $\beta = 2$ - complex numbers (real at the diagonal!)

c) **symplectic ensemble**, $\beta = 4$ - quaternions (2×2 matrices)
leading to $2N \times 2N$ matrix with each eigenvalue occurring twice.

Different normalization conditions, we use the one implied by
the **normal distribution** $\langle H_{ij} \rangle = 0$ and $\sigma^2 = \langle H_{ij}^2 \rangle = 1$

GUE $\Rightarrow$ Unitary invariance, $P(H) = P(UHU^\dagger)$

leads to **j**oint **p**robability **d**istribution **(jpd)** of eigenvalues x_i

$$P_\beta(x_1, \ldots, x_N) = C_N e^{-\frac{\beta}{2}\sum_j x_j^2} \prod_{j<k} |x_j - x_k|^\beta$$

(a general expression for all three ensembles, $\beta = 1, 2, 4$)

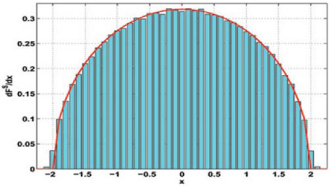
Universal behaviour: Wigner Semicircle Law

Spectral density $P(x)$ for random hermitian matrices

can be obtained by integrating out all eigenvalues but one from jpd.
For all three **Gaussian ensembles** of Hermitian random matrices one obtains (asymptotically, for $N \to \infty$) the **Wigner Semicircle Law (1955)**

$$P(x) = \frac{1}{2\pi}\sqrt{2 - x^2}$$

where x denotes a **normalized eigenvalue**, $x_i = \lambda_i/\sqrt{N}$



Normalised eigenvalue distribution of a random 100 × 100 GUE matrix. (Image by Alan Edelman.)

Random Matrices & Universality

Universality classes

Depending on the symmetry properties of the system we use ensembles form

orthogonal ($\beta = 1$), (anti-unitary symmetry)
unitary ($\beta = 2$), (no anti-unitary symmetry)
symplectic ($\beta = 4$), (symmetry + half-integer spin)
universality class.

The exponent β determines the level repulsion,

$$P(s) \sim s^{\beta}$$

for $s \to 0$ where s stands for the (normalised) level spacing, $s_i = \phi_{i+1} - \phi_i$.

see e.g. **Fritz Haake**, *Quantum Signatures of Chaos*

Level spacing distribution $P(s)$

Nearest neighbour spacing 's'

$s_i = \frac{x_{i+1}-x_i}{\Delta}$ ("s" of Wigner), where Δ is the mean spacing
a) Gaussian ensembles for $N = 2$ $\Rightarrow$ **Wigner surmise**

- $\beta = 1$ **GOE** (orthogonal) $P_1(s) = \frac{\pi}{2} s \exp(-\frac{\pi}{4} s^2) \sim s^1$
- $\beta = 2$ **GUE** (unitary) $P_2(s) = \frac{32}{\pi^2} s^2 \exp(-\frac{4}{\pi} s^2) \sim s^2$
- $\beta = 4$ **GSE** (symplectic) $P_4(s) = \frac{2^{18}}{3^6 \pi^3} s^4 \exp(-\frac{64}{9\pi} s^2) \sim s^4$.

These distributions derived for $N = 2$ work well also for Gaussian ensembles in the asymptotic case, $N \to \infty$.

Random unitary matrices & Circular ensembles of Dyson

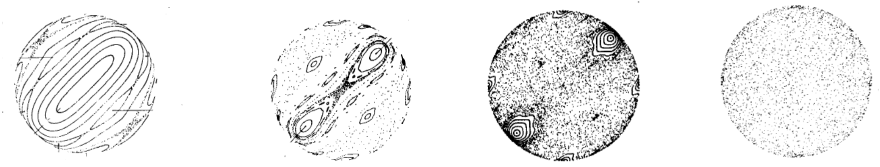
Uniform density of phases along the unit circle, $P(\phi) = 1/2\pi$.
Phase spacing, $s_i = \frac{N}{2\pi}[\phi_{i+1} - \phi_i]$ since $\Delta = 2\pi/N$.
For large matrices the **level spacing** distributions for **Gaussian ensembles** (Hermitian matrices) and **circular ensembles** (unitary matrices) coincide.

Classical *kicked top* model - Haake, Kuś, Scharf 1987

Discrete dynamics on a sphere: $X^2 + Y^2 + Z^2 = 1$

$$X' = \mathrm{Re}(X\cos p + Z\sin p + iY)e^{ikZ\cos p - X\sin p},$$
$$Y' = \mathrm{Im}(X\cos p + Z\sin p + iY)e^{ikZ\cos p - X\sin p},$$
$$Z' = -X\sin p + Z\cos p.$$

linear rotation parameter: $p = \pi/2$,



kicking strength k

$k = 2.0$ $k = 2.5$ $k = 3.0$ $k = 6.0$

transition to chaos: increase of the **Lapunov exponent** λ and **Kolmogorov–Synai** dynamical entropy H_{KS}.

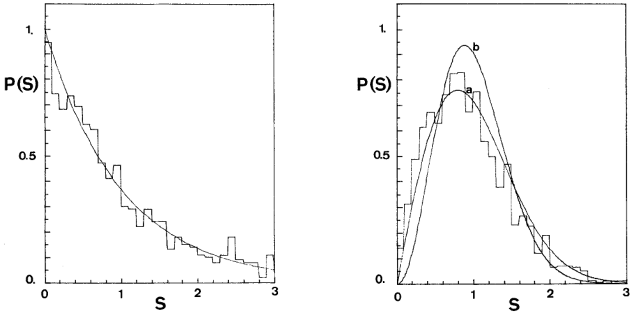
Quantum *kicked top* - Haake, Kuś, Scharf 1987

Discrete dynamics in Hilbert space of dimension $N = 2j + 1$

i) **Hamiltonian** $H(t) = pJ_y + \frac{k}{2j} J_z^2 \sum_{j=-\infty}^{+\infty} \delta(t-n)$

ii) **Unitary evolution operator** $U = \exp[-i(k/2j)J_z^2] \exp[-ipJ_y]$

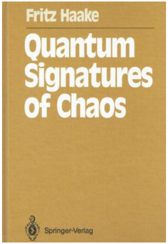
Level spacing distribution $P(s)$ (where $s = (\phi_{i+1} - \phi_i)N/2\pi$)

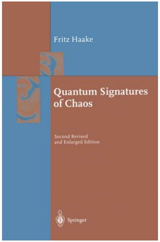


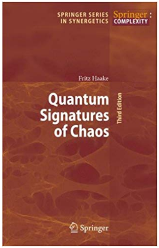
a) $k \in [0.1, 0.3]$ (regular dynamics) b) $k \in [10.0, 10.5]$ (chaotic dynamics)

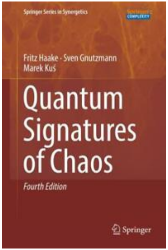
$N = 201$

Four editions (1991 – 2018) of the key reference on **quantum chaos**









Quantum unitary *kicked top* – recapitulation

a) In the case of **classically regular** motion the eigenvalues are **not** correlated, so level spacing distribution $P(s)$ displays level **clustering**, (**Poisson** distribution)

b) In the case of **classically chaotic** motion the eigenvalues **are** correlated, so level spacing distribution $P(s)$ displays level **repulsion**, (**Wigner** distribution) with the repulsion exponent β determined by the symmetry class.

For a suitable choice of parameters **deterministic** unitary evolution matrices $U = U(p, k)$ display statistical properties of **random matrices** of **circular unitary** ensemble — their eigenvectors are **delocalized**.

Bohigas, Giannoni, Schmit (BGS) conjecture (1984):

Under condition of **classical chaos** *deterministic* quantum evolution operators are described by **random matrices**.

Wawel castle in Cracow

Danuta & Krzysztof Ciesielscy theorem

D.& K. Ciesielscy theorem: For any $\epsilon > 0$ there exist $\eta > 0$ such that with **probability** $1 - \epsilon$ the bench **Banach** talked to **Nikodym** in **1916** was localized in η-**neighbourhood** of the **red arrow**.

Plate commemorating the discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Quantum Chaos & Complexity I

Observation: *deterministic* quantum **chaotic** systems,

say: $U = \exp[-i(10/2j)J_z^2]\exp[-i\pi J_y/2]$, (chaotic kicked top)

(**low Kolmogorov complexity**)

display statistical properties typical of ensembles of *random matrices*

(**high Kolmogorov complexity**).

How to define **complexity** for **quantum** dynamics?

OTOC (out-ot-time-order correlations)

$$F(t) := \langle A^\dagger(t)B^\dagger(0)A(t)B(0)\rangle$$

$$C(t) := \langle [A(t), B(0)]^2\rangle = 2[1 - \mathrm{Re}(F(t))]$$

Larkin and Ovchinnikov (1969); Maldacena, Shenker, Stanford (2016)

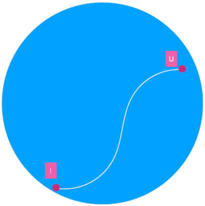
geometric complexity,

Nielsen (2005)

– length of the shortest geodesic

from $\mathbb{I}$ to U

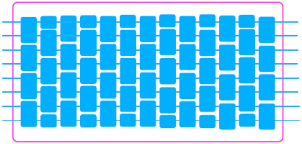
figure by L. Li et al. 2025



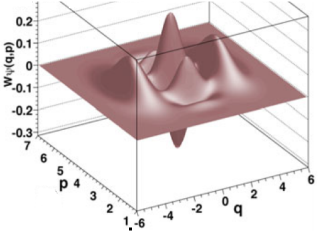
Quantum Chaos & Complexity II

holographic complexity, Susskind (2016)

circuit complexity –
minimal number of **elementary gates**
used to produce unitary $U =$
Jefferson and Myers (2017)
figure by L. Li et al. 2025



Krylov complexity (operator growth), Parker (2019); Caputa (2021)



Kenfack and K. Ż. *Negativity of the Wigner function as an indicator of nonclassicality* (2004)

quantum **complexity**
defined by growth of
Wigner negativity
(defined for discrete Hilbert space)

$$\mathcal{N}(\rho) = \sum_{p,q} |W_\rho(p,q)|$$

Basu, Chowdhury, Ganguly, Nath, Parrikara, Paul (2025)

Wasserstein Complexity

Wasserstein complexity of quantum circuits (and quantum channels)
Li, Bu, Koh, Jaffe, Lloyd (2025)

For a quantum gate U they define its **Wasserstein complexity** C_W,

$$C_W(U) = \max_\psi D_W\Big(|\psi\rangle, U|\psi\rangle\Big),$$

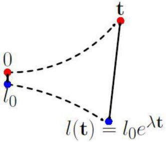
where D_W stands for the **Wasserstein** distance
(Monge-Kantorovich *earth mover* distance, to be defined below)
which is not **unitary invariant** !

$$D_W(|\psi\rangle, |\phi\rangle) \neq D_W(U|\psi\rangle, U|\phi\rangle),$$

in contrast to standard distances D_s (Hilbert-Schmidt, Bures, trace)
for which relation: $D_s(|\psi\rangle, |\phi\rangle) = D_s(U|\psi\rangle, U|\phi\rangle)$ holds.

Wasserstein distance D_W has several appealing properties,
but it is not easy to evaluate!

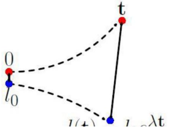
How to define a quantum analogue of the **Lyapunov exponent** ?



but $D_{HS}(\rho, \sigma) = D_{HS}(U\rho U^\dagger, U\sigma U^\dagger)$.

any unitary dynamics
does not change the standard distances !

How to define a quantum analogue of the **Lyapunov exponent** ?



but $D_{HS}(\rho,\sigma) = D_{HS}(U\rho U^\dagger, U\sigma U^\dagger)$.

any unitary dynamics

does not change the standard distances !

Ann. Physik 1 (1992) 531 – 539

At the basis of our study lies a generalization of the Lyapunov exponent,

$$\lambda = \lim_{t\to\infty} \lambda(t), \quad \lambda(t) = \lim_{d(0)\to 0} \frac{1}{t} \ln\left(\frac{d(t)}{d(0)}\right),$$

Annalen der Physik
Johann Ambrosius Barth 1992

which distance d ?

Lyapunov exponents from quantum dynamics

Fritz Haake, Harald Wiedemann, and Karol Życzkowski*

Vistas in Astronomy, Vol. 37, pp. 153–156, 1993
0083–6656/93 $24.00

HOW TO GENERALIZE THE LAPUNOV EXPONENT FOR QUANTUM MECHANICS

Karol Życzkowski, *† Harald Wiedemann†
and Wojciech Słomczyński‡

Monge distance between both Q - functions

Are all 'reasonable' distances between quantum states unitarily invariant,

$$D(\rho,\sigma) = D(U\rho U^\dagger, U\sigma U^\dagger) \ ?$$

a counter example: the **Monge distance**

J. Phys. A: Math. Gen. **31** (1998) 9095–9104. Printed in the UK PII: S0305-4470(98)93137-7

The Monge distance between quantum states

Karol Życzkowski†§ and Wojeciech Słomczyński‡‖

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 6689–6722 PII: S0305-4470(01)18080-7

The Monge metric on the sphere and geometry of quantum states

Karol Życzkowski[1,2] and Wojciech Słomczyński[3]

defined between the corresponding Q-functions, $Q_i(\alpha) = \langle\alpha|\rho_i|\alpha\rangle$,

$$D_M(\rho_1,\rho_2) = D_M\big(Q_1(\alpha), Q_2(\alpha)\big)$$

Monge problem (1781)

An optimal scheme of translocation of soil between the initial shape $Q_1(x_1, x_2)$ and the final one $Q_2(x_1, x_2)$ gives the **Monge distance** between both probability distributions, $D_M(Q_1, Q_2)$.

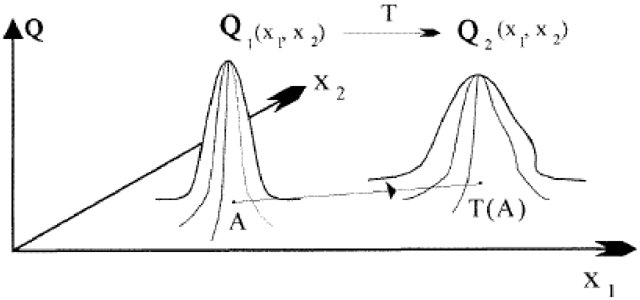
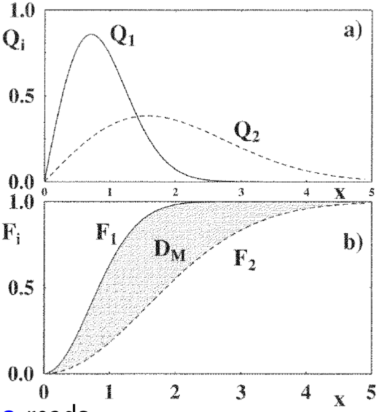


Figure 1. Monge transport problem: how to move a pile of sand $Q_1(x_1, x_2)$ to a new location $Q_2(x_1, x_2)$ minimizing the work done?

minimize the **total work** against **friction**, (neglect the vertical component)

1D problem – solution of T. Salvemini
Sul calcolo degli indici di concordanza... (1943)

For any two 1D probability distributions $Q_1(t)$ and $Q_2(t)$,
represented by their cummulative distributions, $F_i(x) = \int_{-\infty}^{x} Q_i(t)\, dt$,



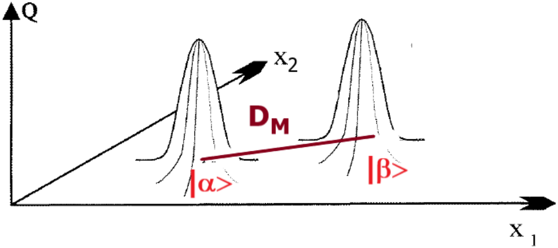
their **Monge distance** reads,

$$D_M(Q_1, Q_2) = \int_{-\infty}^{+\infty} |F_1(x) - F_2(x)|\, dx.$$

Monge metric & quantum states: a) infinite space

natural choice: harmonic oscillator **coherent states** $|\alpha\rangle$ for $\alpha \in \mathbb{C}$

Monge distance between any two *coherent states* satisfies classical property :



$$D_M(|\alpha\rangle, |\beta\rangle) = |\alpha - \beta|$$

2D problems with radial symmetry $\Rightarrow$ **1D** solution of Salvemini works!

Fock states $|n\rangle$ with $n = 0, 1, 2 \ldots$ *with* $D_{HS}(|i\rangle, |j\rangle) = \sqrt{2} = \text{const}$

$D_M(|0\rangle, |1\rangle) << D_M(|1\rangle, |100\rangle)$ (*as desired*)

thermal states $|\bar{n}\rangle$ with mean number of photons equal to $\bar{n}$

$D_M(|\bar{n}\rangle, |\bar{m}\rangle) \approx |\sqrt{n} - \sqrt{m}|.$

Wawel Castle in Cracow

transport problem – **Kantorovich** formulation (1939)

Mathematical Methods in the Organization and Planning of Production

Transport plan

A **transport plan** is a measure ω on $X \times Y$ such that

$$\omega(A \times Y) = \mu(A), \; \omega(X \times B) = \nu(B), \text{ for any } A \subset X, \; B \subset Y.$$

Kantorovich optimal transport problem (1942)

Denote by $\Gamma(\mu, \nu)$ the set of all transference plans for fixed μ, ν.

$$\text{Find } \gamma, \text{ which realises } \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} c(x,y) d\gamma(x,y).$$

Wasserstein p–distances (1969) (classical)

Let $Y = X$ and take c to be a **distance function**. Then, for any $p \geq 1$,

$$W_{c,p}(\mu, \nu) := \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X \times Y} c(x,y)^p d\gamma(x,y) \right)^{1/p}$$

is a distance on $\mathcal{P}(X)$.

Discrete optimal transport

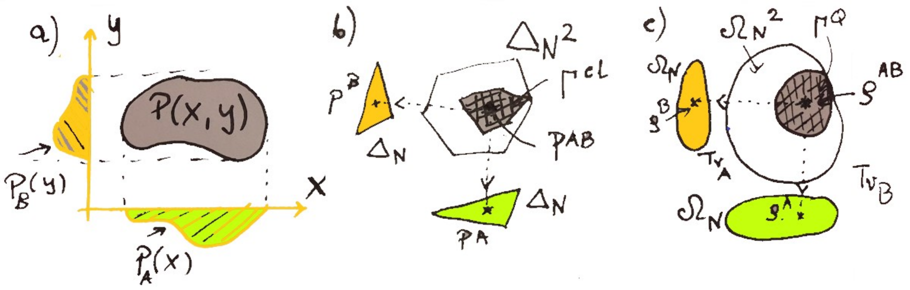
- Take an N point set $X = Y = \{x_i\}_{i=1}^N$.
- Consider two probability vectors p^A, p^B of length N,
  which can be seen as *classical states* $p^A, p^B \in \mathcal{P}(X)$.
- A transport plan $P^{AB} \in \Gamma^{cl}(p^A, p^B)$ is a classical state $\mathcal{P}(X \times X)$.
- P^{AB} is identified with the probability vector $\widetilde{P}^{AB}$ of length N^2.
- Define a *diagonal* **coupling matrix** $\rho^{AB}_{\mu\nu} := \widetilde{P}^{AB}_{\mu}\delta_{\mu\nu}$, for $\mu, \nu = 1, \ldots, N^2$.

- Take a distance function d on X and define a matrix $E_{ij} := d(x_i, x_j)$.
- Recast E into a vector $\widetilde{E}$ of length N^2.
- Define a *diagonal* **cost matrix** $C^{cl}_{\mu\nu} := \widetilde{E}_{\mu}\delta_{\mu\nu}$.
- The **classical optimal transport problem** then reads

$$T^{cl}_C(p^A, p^B) := \min_{P^{AB} \in \Gamma^{cl}(p^A, p^B)} \operatorname{Tr} C^{cl} \rho^{AB}.$$

Quantum optimal transport – idea

Kantorovich formulation of transport problem for:



a) continuous 1D probabilities $p_A(x)$ and $p_B(y)$
coupled by a joint distribution $P(x, y)$;

b) two N-point classical states $p^A, p^B \in \Delta_N$ coupled by a joint state
$P^{AB} \in \Gamma^{cl} \subset \Delta_{N^2}$ with adjusted marginals;

c) two quantum states $\rho^A, \rho^B \in \Omega_N$ coupled by a bipartite state
$\rho^{AB} \in \Gamma^Q \subset \Omega_{N^2}$ such that $\mathrm{Tr}_A\, \rho^{AB} = \rho^B$ and $\mathrm{Tr}_B\, \rho^{AB} = \rho^A$.

Quantum optimal transport – brief history

- Monge problem for Husimi distributions of quantum states.
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- Dynamical formulation [Benamou–Brenier (2000)].
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  - N. Datta, C. Rouzé, *Ann. H. Poincaré* **21**, 2115 (2020).
  - K. Ikeda, *Quantum Inform. Process.* **19**, 25 (2020).

- Direct generalisations using quantum couplings
  - F. Golse, C. Mouhot, T. Paul, *Commun. Math. Phys.* **343**, 165 (2016).
  - M.H. Reira, *Bachelor's Thesis Universitat Autònoma de Barcelona* (2018).
  - N. Yu, L. Zhou, S. Ying, M. Ying, *arXiv*:1803.02673 (2018).
  - S. Chakrabarti, Y. Huang, T. Li, S. Feizi, X. Wu, *arXiv*:1911.00111 (2019).
  - G. De Palma, D. Trevisan, *arXiv*:1911.00803 (2019).
  - E. Caglioti, F. Golse, T. Paul, *J. Stat. Phys.*, **181**, 149 (2020).
  - G. De Palma, M. Marvian, D. Trevisan, S. Lloyd, IEEE Trans. Inf. Theor. (2021)
  - R. Duvenhage, J. Operator Theory (2022).
  - Friedland, Eckstein, Cole, K. Ż. *Phys. Rev. Lett.* (2022)
  - *several other recent papers,* (2022-2025)

Quantum optimal transport – definition

- $\Omega_N := \{\rho \in \mathcal{B}(\mathbb{C}^N) \mid \rho = \rho^\dagger, \ \rho \geq 0, \ \mathrm{Tr}\,\rho = 1\}$ density matrices of order N.
- Fix two states $\rho^A, \rho^B \in \Omega_N$.
- Consider a **coupling matrix** (or "**quantum transport plan**")
  $$\rho^{AB} \in \Omega_{N^2}, \text{ such that } \mathrm{Tr}_A\,\rho^{AB} = \rho^B \text{ and } \mathrm{Tr}_B\,\rho^{AB} = \rho^A.$$
- Denote by $\Gamma^Q(\rho^A, \rho^B) \subset \Omega_{N^2}$ the set of all coupling matrices.
  - Note that $\rho^A \otimes \rho^B \in \Gamma^Q(\rho^A, \rho^B)$.
- Take a **quantum cost matrix** $C = C^\dagger \in \mathcal{B}(\mathbb{C}^{N \times N})$.
- The **quantum optimal transport problem** defined by the **minimum**
  $$T_C^Q(\rho^A, \rho^B) := \min_{\rho^{AB} \in \Gamma^Q(\rho^A, \rho^B)} \mathrm{Tr}\, C\rho^{AB}.$$

How to select a suitable **cost matrix C**?

Quantum cost matrix C^Q: $\mathrm{diag}(C^Q) = C^{cl}$.

Motivations:

- semi-classical limit of QM (∞ dim) [Golse, Mouhot, Paul, Caglioti]
- quantum transport plans $\leftrightarrow$ quantum channels [De Palma, Trevisan (2019)]
- Hamming distance [De Palma, Marvian, Trevisan, Lloyd (2019)]

Our motivation: (coherification of the diagonal classical cost matrix C^{cl})

- Find cost matrices, which yield an analogue of **Wasserstein** distances.

Projective cost matrix C^Q – antisymmetric subspace – singlet state

Take a computational basis $\{|i\rangle\}_{i=1}^N$ and set $|\psi_{ij}^-\rangle = \frac{1}{\sqrt{2}}(|i,j\rangle - |j,i\rangle)$.

$$C^Q = \sum_{j>i=1}^{N} |\psi_{ij}^-\rangle\langle\psi_{ij}^-| = \tfrac{1}{2}\big(\mathbb{1}_{N^2} - \mathrm{SWAP}\big) = (C^Q)^2.$$

The same idea explored in: Reira (2018); Yu, Zhou, Ying, Ying (2018) and Chakrabarti, Huang, Li, Feizi, Wu (2019).

p–Wasserstein distances

p-Wasserstein distances

If the classical cost matrix comes from a distance on X then, in analogy to the p–norm, for any $p \geq 1$ one defines,

$$W^{cl}_{C,p}(p^A, p^B) := \left(T^{cl}_{C^p}(p^A, p^B)\right)^{1/p} = \left(\min_{P^{AB} \in \Gamma^{cl}(p^A, p^B)} \mathrm{Tr}\, C^p \rho^{AB}\right)^{1/p}$$

is a distance on $\mathcal{P}(X)$.

<u>Remark</u>:

- If X has the geometry of a *simplex*, i.e. $d(x_i, x_j) = 1 - \delta_{ij}$, then $C^p = C$ and $W^{cl}_{C,p} = (W^{cl}_{C,1})^{1/p}$ for any $p \geq 1$.

Bounds on quantum optimal transport, $W = \sqrt{T^Q}$

Fidelity $F(\rho^A, \rho^B) := \left(\mathrm{Tr}\left|\sqrt{\rho^A}\sqrt{\rho^B}\right|\right)^2.$

Quantum distances:

$$I := \sqrt{1 - F}, \qquad \text{root infidelity,}$$

$$B := \sqrt{2\left(1 - \sqrt{F}\right)} \qquad \text{Bures distance.}$$

Bounds on quantum optimal transport, $W = \sqrt{T^Q}$

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Quantum distances:

$I := \sqrt{1 - F},$ root infidelity,

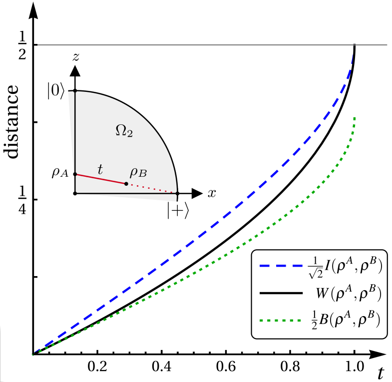
$B := \sqrt{2\left(1 - \sqrt{F}\right)}$ Bures distance.

Theorem: bounds for $W = \sqrt{T^Q}$
(based on [Yu, Zhou, Ying, Ying (2018)])

For any $\rho^A, \rho^B \in \Omega_N$ we have

$$\frac{1}{\sqrt{2}} I(\rho^A, \rho^B) \geq W(\rho^A, \rho^B) \geq \frac{1}{2} B(\rho^A, \rho^B).$$

Left ineqality is saturated
if ρ^A or ρ^B is pure.



comparison of distances for an exemplary trajectory

$$\rho^A = \frac{9}{20}\mathbb{1} + \frac{1}{10}|0\rangle\langle 0|,$$
$$\rho^B = (1-t)\rho^A + t(|+\rangle\langle +|)$$

Transport metric on the Bloch ball

Theorem

For $N = 2$, W_p satisfies the **triangle inequality** iff $p \geq 2$:
For all $\rho^A, \rho^B, \rho^C \in \Omega_2$

$$W_p(\rho^A, \rho^B) + W_p(\rho^B, \rho^C) \geq W_p(\rho^A, \rho^C).$$

Thus, W_p for $p \geq 2$ forms a **distance** on the **Bloch ball** Ω_2.

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- W_p is *not* a distance for $p < 2$, so the triangle inequality fails for T^Q, but it holds for $W = W_2 = \sqrt{T^Q}$

Computation of T^Q on Ω_2 amounts to solving a 6th order polynomial eq.

- For *classical* states $\rho_r^{cl} = \mathrm{diag}(r, 1-r)$, we obtain

$$W\left(\rho_r^{cl}, \rho_s^{cl}\right) = \frac{1}{\sqrt{2}} \max\left\{\left|\sqrt{r} - \sqrt{s}\right|, \left|\sqrt{1-r} - \sqrt{1-s}\right|\right\}.$$

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- For *isospectral* states, $\mathrm{eig}(\rho) = \{\lambda, 1-\lambda\}$ and $U = O(\theta) \in \mathcal{U}(2)$,
$$W(\rho, U\rho U^\dagger) = \sqrt{\frac{1}{\sqrt{2}} - \sqrt{\lambda(1-\lambda)}}\,\left|\sin(\theta/2)\right|.$$

Geodesic lines

Are there $\rho^A, \rho^B, \rho^C \in \Omega_2$ such that

$$W(\rho^A, \rho^B) + W(\rho^B, \rho^C) = W(\rho^A, \rho^C) \quad ?$$

Friedland, Eckstein, Cole, K. Ż. *Phys. Rev. Lett.* **2022**

Transport metric on the Bloch ball – geodesics

Geodesic lines

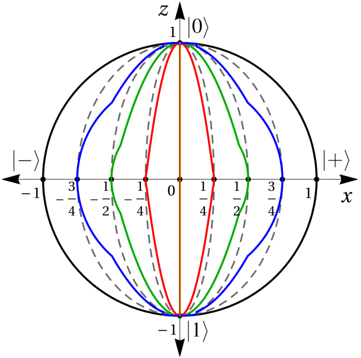
Are there $\rho^A, \rho^B, \rho^C \in \Omega_2$ such that

$$W(\rho^A, \rho^B) + W(\rho^B, \rho^C) = W(\rho^A, \rho^C) \quad ?$$

- No geodesics for root infidelity I and Bures distance B.
- But there are geodesics for the **Bures angle**

$$A(\rho^A, \rho^B) := \tfrac{2}{\pi} \arccos \sqrt{F(\rho^A, \rho^B)}$$

- ...and there are geodesics for the transport metric W!



Friedland, Eckstein, Cole, K. Ż. *Phys. Rev. Lett.* **2022**

Quantum vs classical optimal transport

- **Decoherence**: $\rho_\alpha := \alpha\rho + (1-\alpha)\,\mathrm{diag}(\rho)$ for $\alpha \in [0,1]$.
  - ρ_0 is a classical state
  - α is proportional to the l_1-coherence of ρ_α

Quantum vs classical optimal transport

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The transport of quantum states is more expensive

The optimal quantum transport cost between two density matrices $\rho_\alpha^A \neq \rho_\alpha^B \in \Omega_N$ decreases with the parameter α,

$$T^Q(\rho_\alpha^A, \rho_\alpha^B) \leq T^Q(\rho_\beta^A, \rho_\beta^B), \quad \text{for} \quad 0 \leq \alpha \leq \beta \leq 1.$$

Quantum vs classical optimal transport

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Quantum optimal transport is cheaper (cf. [Caglioti, Golse, Paul (2020)])

Let $\mathbf{r}, \mathbf{s}$ be N-dim probability vectors and let $\rho_{\mathbf{r}}^{cl}, \rho_{\mathbf{s}}^{cl} \in \Omega_N$ be the corresponding quantum states defined as $(\rho_{\mathbf{p}}^{cl})_{ij} := p_i\delta_{ij}$.

Then, with $C^{cl} = \mathrm{diag}(C^Q)$,

$$T^Q\left(\rho_{\mathbf{r}}^{cl}, \rho_{\mathbf{s}}^{cl}\right) \;\leq\; T^{cl}\left(\mathbf{r}, \mathbf{s}\right).$$

Quantum-to-classical transition for transport cost

- **Cost matrix decoherence**: $C^Q_\alpha := \alpha C^Q + (1-\alpha)\,\text{diag}(C^Q)$, $\alpha \in [0,1]$.
  - C^Q_0 is a classical cost matrix
  - α is proportional to the l_1-coherence of ρ_α
  - C^Q_α is *not* a quantum cost matrix for $\alpha < 1$.

Quantum-to-classical transition for transport cost

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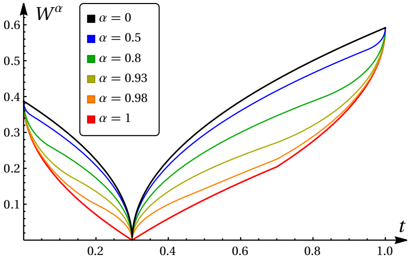
  - C^Q_0 is a classical cost matrix
  - α is proportional to the l_1-coherence of ρ_α
  - C^Q_α is *not* a quantum cost matrix for $\alpha < 1$.

- Define $W^\alpha := \sqrt{T^Q_\alpha}$ with

$$T^Q_\alpha(\rho^A, \rho^B) := \min_{\rho^{AB} \in \Gamma^Q} \left(\mathrm{Tr}\; C^Q_\alpha \rho^{AB} \right).$$

- Take classical states of order 2: $\rho^{cl}_r = \mathrm{diag}(r, 1-r)$.
- $W^\alpha(\rho^{cl}_r, \rho^{cl}_s)$ is a **strictly decreasing** function of α, unless either of states is pure.



$W^\alpha(\mathrm{diag}(0.3, 0.7), \mathrm{diag}(t, 1-t))$

Energy distance for pure quantum states, $N \geq 2$

1. **Monge distance** defined by coherent states is not easy to compute... hard **optimization problem** (*even for two pure states*)

2. For pure states the **Wasserstein distance** determined by any classical Euclidean distance matrix $E_{ij} = d(x_i, x_j)$ is given **explicitely** !

Example - N points on an (energy) **line**: $E_{ij} = d(x_i, x_j) = |x_i, x_j|$

For a *given* **Hamiltonian** H with non-degenerate eigenvalues E_i and eigenvectors $|i\rangle$, so that $H|i\rangle = E_i|i\rangle$, we set $E_{ij} = |E_i - E_j|$ and obtain

$$W_H^2(|\psi\rangle, |\phi\rangle) = \sum_{j>i=1}^{N} |E_i - E_j|^2 \, |\psi_i\phi_j - \phi_i\psi_j|^2,$$

where the analyzed states are expanded in eigenbasis of Hamiltonian, $|\psi\rangle = \sum_i \psi_i|i\rangle$ and $|\phi\rangle = \sum_j \phi_j|j\rangle$.

Energy distance determined by a Hamiltonian H

1. **Energy distance** for any two eigenstates of H are equal to the energy difference

$$W(|i\rangle, |j\rangle) = |E_i - E_j| \quad (**)$$

2. For any to pure states $|\psi\rangle$ and $|\phi\rangle$ their **Energy distance** satisfies the bounds

$$|\langle\phi|H|\phi\rangle - \langle\phi|H|\phi\rangle|^2 \leq W^2(|\phi\rangle, |\psi\rangle) \leq |\langle\phi|H|\phi\rangle - \langle\phi|H|\phi\rangle|^2 + 2(\Delta_\phi^2 + \Delta_\psi^2)$$

where the variance read $\Delta_\phi^2 = \langle\phi|H^2|\phi\rangle - \langle\phi|H|\phi\rangle^2$.

which for two eigenstates ($\Delta_\phi = \Delta_\psi^2 = 0$) implies Eq. (**).

Example: 1D **Hydrogen atom**, $H = p^2/2m - e^2/r$ and its eigenstates $|n\rangle$: any *standard distance* D_x (trace, HS, Bures) imply equilateral triangle,

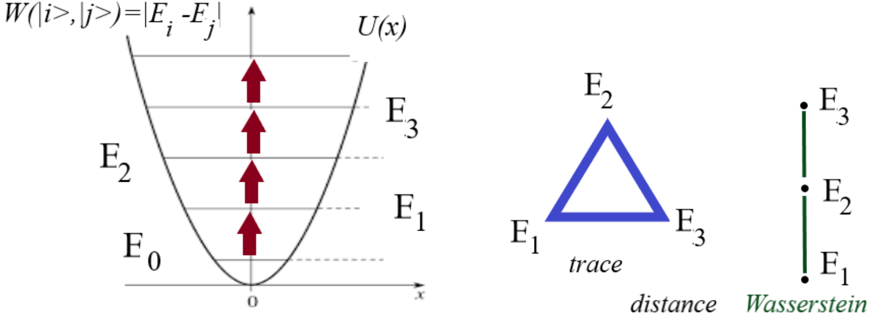
$$D_x(|0\rangle, |1\rangle) = D_x(|1\rangle, |100\rangle) = D_x(|0\rangle, |100\rangle) \text{ for all eigenstates,}$$

while the **energy** (**Wasserstein**) distance reveals the energy difference:

$$W(|0\rangle, |1\rangle) << W(|1\rangle, |100\rangle) < D_x(|0\rangle, |100\rangle).$$

Trace distance & Energy distance

For eigenstates of H the *energy distance* is equal to the number of **resonant photons** absorbed during the transition



In such a case the **trace distance** between orthogonal states forms an **equilateral triangle**, $D_{tr}(|1\rangle, |3\rangle) = D_{tr}(|1\rangle, |2\rangle) = D_{tr}(|2\rangle, |3\rangle)$,

while the **Energy distance** forms a **metric line**'

$$W(|1\rangle, |3\rangle) = W(|1\rangle, |2\rangle) + W(|2\rangle, |3\rangle).$$

Angular momentum distance W_J

Example 2: **Angular momentum** operator, $J^2 = J_x^2 + J_y^2 + J_z^2$

If $H = J = \sqrt{J^2}$ then the corresponding **angular momentum** distance W_J satisfies the **semiclassical** property:

Let $|0\rangle = |j, j\rangle$ be the maximum weight state (north pole!) and the state $|\theta\rangle := \exp(-i\theta J_y)|0\rangle$ denote **spin coherent** state pointing in direction θ.

Then, semiclassically, $j >> 1$, the Wasserstein distance W_J between both coherent states tends to the geodesic distance between both points at the sphere:

$$W_J(|0\rangle, |\theta\rangle) \;\approx\; j\,\theta$$

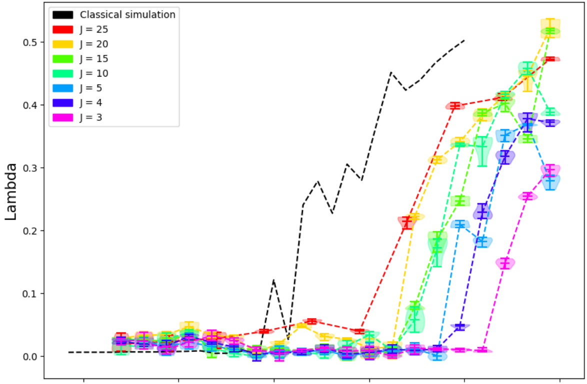
as desired for analyzis of the semiclassical regime $j >> 1$.

Main idea of this application:

For a given unitary matrix $U \in U(N)$ and originally close coherent states $|\alpha(0)\rangle$, $|\beta(0)\rangle$, analyze, how the angular momentum distance between them evolves in time, $W_J(t) = W_J\big(|\alpha(t), \beta(t)\rangle\big)$

Novel attempt to define quantum Lapunov exponent

Transition to chaos: Classical and quantum dynamics of **kicked top**.
Phase space approach: placing two coherent states at two points close to a selected point (θ_0, ϕ_0) we study the diveregence of the **Wasserstein** distance $W_J(t)$ and evaluate Lapunov exponents Λ (*a signature of quantum complexity*) for different dimensions $N = 2j + 1 = [7, ...51]$ and kicking strength $k \in [1, 3.6]$.



$k \in [1, 3.6]$.

Quantization of a classical distance: a general approach

Consider a set of N points $x_i \in \mathbb{R}^m$, $k = 1, \ldots, N$.
Denote distances between them by $d_{ij} = d(x_i, x_j)$, also not Euclidean !

Theorem: (**Bistroń, Miller**, 2025 *to appear*). For any chosen classical distance matrix $d_{ij} = d_{ji} \geq 0$ of order N, the map acting on the space of pure quantum states of size N,

$$D_W^2(|\psi\rangle, |\phi\rangle) := \sum_{j>i=1}^N d_{ij}^2 \; |\psi_i\phi_j - \phi_i\psi_j|^2,$$

satisfies the triangle inequality and induces a **quantum distance** in the complex projective space $\mathbb{C}P^{N-1}$.

Here ψ_i and ϕ_j denote complex expansion coefficients,
$|\psi\rangle = \sum_{i=1}^N \psi_i |i\rangle$ and $|\phi\rangle = \sum_{j=1}^N \phi_j |j\rangle$.

Proof is based on a generalized Cauchy - Schwarz inequality

generalized **Cauchy - Schwarz** inequality (complex case),
(coefficients ω_{ijk} can be negative!)

Theorem 1. *Fix $n \geq 3$ and an orthonormal system $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \subset \mathbb{C}^n$, and define ω_{ijk} as*

$$\omega_{ijk} := \overline{x}_i \overline{y}_j \overline{z}_k \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix}. \tag{1}$$

Then for any symmetric matrix $(A_{ij}) \in \mathbb{M}_n(\mathbb{R})$

$$\left| \sum_{ijk} A_{ik} A_{jk} \omega_{ijk} \right| \leq \sqrt{\sum_{ijk} A_{ik}^2 \omega_{ijk}} \sqrt{\sum_{ijk} A_{jk}^2 \omega_{ijk}}. \tag{2}$$

Without loss of generality, we can assume that $A_{ii} = 0$ for all i.

Rafał Bistroń and **Tomasz Miller** (2025)

Quantum Hamming distance

Consider two pure states of n-qubit system, $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_2^n$ represented by 2^n coeficients, $\psi_{i_1 \dots i_n}$ and $\phi_{j_1 \dots j_n}$.

Find a true **distance** D_H such that for any two states in the computational basis, $|\Psi\rangle = |i_1 i_2 \dots i_n\rangle$ and $|\Psi\rangle = |j_1 j_2 \dots j_n\rangle$ the distance $D_H(|\Psi\rangle, |\Phi\rangle)$ is equal to the **classical Hamming** distance $d_H(i_k, j_k)$ between the bit strings i_k and j_k,
i.e. the minimal number of NOT gates to transform string i_k into j_k.

Related problem was studied by **Chau** (1999); **De Palma, Marvian, Trevisan, Lloyd** (2019); **Girolami, Anza**, Phys Rev. Lett. (2021); **Grudka, Kurzyński, Sajna, Wójcik**^{2}, Phys. Rev. A (2024).

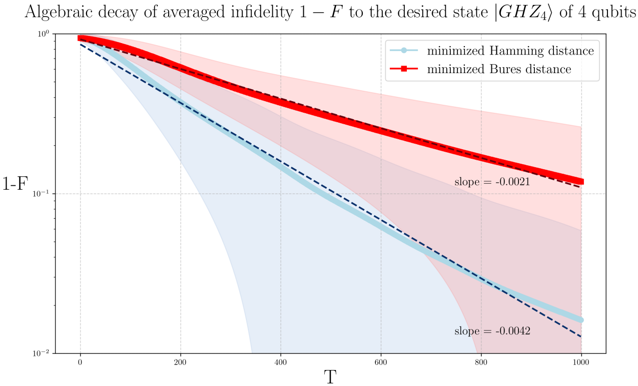
Our explicit solution (no optimization needed!) reads

$$D_H^2(|\psi\rangle, |\phi\rangle) := \sum_{i_1,\dots i_n=0}^{1} d_H^2(i_k, j_k)\ |\psi_{i_k}\phi_{k_j} - \phi_{i_k}\psi_{j_k}|^2,$$

and forms a true distance, as the triangle inequality holds.

Random search procedure: we wish to get close to a given desired state by minimization a *distance* to the goal: 4-qubit state

$$|GHZ_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}.$$



Minimization of **quantum Hamming** distance converges much faster than minimization of *unitarily invariant* **Bures** distance.

other possible application: a measure of **quantum complexity** !

Concluding Remarks

- Several notions of **classical complexity**. Deterministic chaos produces pseudo-random sequences which do not exhibit large **Kolmogorov complexity**.
- There exist several notions of **quantum complexity**: e.g. **geometric** (Nielsen), **holographic** (Susskind), **circuit complexity**.
  Quantized chaotic systems (of low **Kolmogorov complexity**) produce unitary operators with statistical properties characteristic to random matrices (high **complexity**).
- other approaches to describe a given unitary dynamics U involve
  a) evolution of Wigner negativity
  b) maximal Monge-Kantorovich-Wasserstein distance
  between state $|\psi\rangle$ and its image $U|\psi\rangle$.
- Quantum **Monge distance** is not *unitarily invariant*.
  Thus it evolves in time and can be applied to define a notion of **quantum Lyapunov exponent**
  and the notion of quantum *earth mover* (Wasserstein) **complexity**.

Plate commemorating the discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Bench commemorating the discussion between
Otton Nikodym and **Stefan Banach** (**Kraków, summer 1916**)

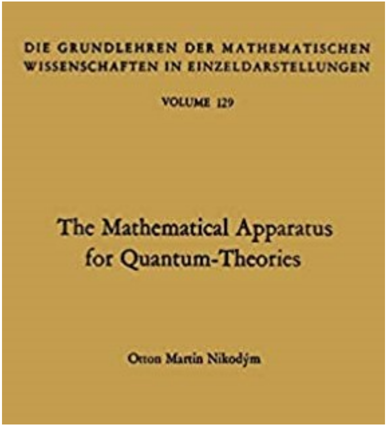
Sculpture: Stefan Dousa Fot. Andrzej Kobos

opened in Planty Garden, Cracow, Oct. 14, 2016

monument of **Stefan Banach** opened in Ostrowsko, **August 31, 2025**

Ostrowsko (Podhale) – family place of Stefan Greczek (father of Banach)

50 years after the discussion at the bench in Cracow,
in 1966, **Otton Nikodym** published the book



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Banach tells his side of the story